

Self-generating Sequences

→ ~~1030~~

→ 1148

In this section are collected a number of sequences which are generated by simple and yet unusual recurrence rules. We have (rather arbitrarily) called them "self-generating". For yet another kind of recurrence rule, see the (next) section, sequences generated by sieves.

In the first two examples let $A = \{a_0 = 1, a_1, a_2, \dots\}$ be a sequence of 1's and 2's.

(1) We obtain a new sequence A' from A by replacing each 1 in A by the finite sequence $U = 1, 2$ and each 2 by $V = 2, 1$.

E.g. For example if $A = \{1, 1, 1, \dots\}$ then

$A' = \{1, 2, 1, 2, 1, 2, \dots\}$. Now impose

the restriction that $A = A'$. ~~This determines~~

~~A uniquely, and we find it is easily~~

~~that~~ $\{1, 2, 2, 1, 2, 1, 1, 2, 2, 1, \dots\}$, which is

sequence 1285.

Similarly with the same U , and $V = 1, 1, 2$ or $1, 2, 2$ we obtain sequences 1030 and 1468 respectively.

(2) ~~Alternately~~ Let $A'' = \{a_0'', a_1'', a_2'', \dots\}$, where a_n'' is the length of the n^{th} run in A . (A run is a maximal string of identical symbols.) ~~Example if $A = \{1, 2, 1, 2, 1, 2, \dots\}$~~

~~Example if $A = \{1, 2, 1, 2, 1, 2, \dots\}$~~ E.g. if $A = \{1, 2, 1, 2, 1, 2, \dots\}$ then $A'' = \{1, 1, 1, 1, 1, 1, \dots\}$. The restriction

that $A = A''$ ~~again determines~~ forces A to be

~~uniquely, and $A = \{1, 2, 2, 1, 1, 2, 1, 2, 2, 1, \dots\}$~~

which is sequence 2.

In the remaining examples $A = \{a_0 = 1, a_1, a_2, \dots\}$ is a ~~monotonically~~ nondecreasing sequence of integers.

(3) Let b_n be the number of times ~~the~~ n occurs in the sequence A , for $n = 1, 2, \dots$. E.g. if $b_n = n$, then $A = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots\}$, sequence

2024. Or if $b_n = a_{n-1}$, then $A = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, \dots\}$, sequence 1462.

In the latter sequence let c_n be the greatest integer m such that $a_{m-1} = n$. Then $\{c_1, c_2, \dots\} = \{1, 3, 5, 8, 11, 15, \dots\}$ is sequence 1463.

$\hookrightarrow 19, 23, 28, 33, \dots$

(4) Choose a_n to be the ^{least} ~~smallest~~ integer such that A contains 1 odd, then 2 even, then 3 odd, then 4 even, $\dots$, numbers. Then $A = \{1, 2, 4, 5, 7, 9, 10, 12, 14, 16, \dots\}$, sequence 1614.

(5) Choose a_{n+1} so that $a_{n+1} - a_n =$ ^{least} ~~smallest~~ positive integer m not of the form $a_i - a_j$ for $0 \leq j < i \leq n$.

Then $A = \{1, 2, 4, 8, 13, 21, \dots\}$, sequence 1148.  

Or choose a_{n+1} so that $a_{n+1} - a_{n-1} = m$, and we obtain sequence 1149.

Choose $a_0 = 1$, $a_1 = 2$, $a_{2n} = 2a_{2n-1}$, and $a_{2n+1} - a_{2n} =$ ~~smallest~~ ^{least} integer positive integer not of the form

$$a_i - a_j \quad \text{for } 0 \leq j < i \leq 2n.$$

Then $A = \{1, 2, 4, 8, 16, 21, \dots\}$, sequence 1856.
 $42, 51, 102, 112,$

Choose $a_0 = 1$, $a_1 = 2$ and $a_n =$ least integer which can be written uniquely as $a_i + a_j$, with $0 \leq i < j \leq n$. Then $A = \{1, 2, 3, 4, 6, 8, 11, 13, 16, \boxed{17}, 18, \dots\}$, sequence 2858. Similarly starting from

~~1, 2, 3, 4, 6, 8, 11, 13, 16, 17, 18, \dots~~

~~1, 2, 3, 4, 6, 8, 11, 13, 16, 17, 18, \dots~~

1, 3 and 2, 3 we obtain sequences ~~2859~~ and 1857 respectively.